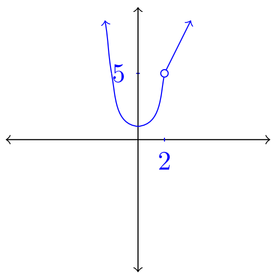
Objectives:

- Determine if a piecewise function is continuous
- State the Intermediate Value Theorem and use to determine information about a function

Continuity practice: Determine where each of these functions is continuous and classify the types of discontinuities. Sketch each graph.

$$f(x) = \begin{cases} x+3 & x > 2 \\ x^2+1 & x < 2 \end{cases}$$

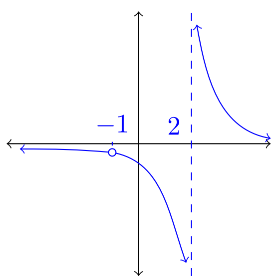


$f(x)$ is defined piecewise with two polynomials which means $f(x)$ is continuous everywhere except possibly $x = 2$.

$f(2)$ is not defined so $f(x)$ is discontinuous there.

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2}(x^2+1) = 5$
$\lim_{x\to 2^+} f(x) = \lim_{x\to 2}(x+3) = 5$
So $\lim_{x\to 2} f(x)$ exists, which means $f(x)$ has a removable discontinuity at $x = 2$.

$$h(x) = \frac{x+1}{x^2-x-2}$$

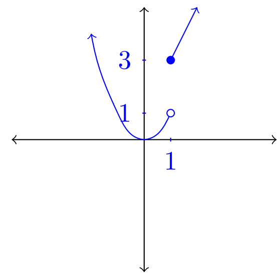


$h(x)$ is a rational function so h is discontinuous only where undefined, at $x = -1$ and $x = 2$.

$\lim\limits_{x\to -1} h(x) = \lim\limits_{x\to -1} \frac{x+1}{(x+1)(x-2)} = \lim\limits_{x\to -1} \frac{1}{(x-2)} = -\frac{1}{3}$
So $h(x)$ has a removable discontinuity at $x = -1$.

$\lim\limits_{x\to 2^+} h(x) = \lim\limits_{x\to 2^+} \frac{1}{x-2} = \infty$ (think: "$\frac{1}{\text{tiny pos.}}$")
$\lim\limits_{x\to 2^-} h(x) = \lim\limits_{x\to 2^-} \frac{1}{x-2} = -\infty$ (think: "$\frac{1}{\text{tiny neg.}}$")
So $h(x)$ has an infinite discontinuity at $x = 2$ (a vertical asymptote).

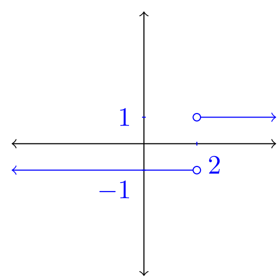
$$g(t) = \begin{cases} 2t+1 & t \geq 1 \\ t^2 & t < 1 \end{cases}$$



$g(t)$ is defined piecewise with two polynomials which means $g(t)$ is continuous everywhere except possibly $t = 1$.

$\lim_{t\to 1^-} g(t) = \lim_{t\to 1} t^2 = 1$
$\lim_{t\to 1^+} g(t) = \lim_{t\to 1}(2t+1) = 3$
So $\lim_{t\to 1} g(t)$ does not exist. But, since the right and left limits exist but differ, $g(t)$ has a jump discontinuity at $t = 2$.
Since $\lim_{t\to 1^+} g(t) = g(1)$, $g(x)$ is right-continuous at $t = 1$.

$$r(u) = \frac{|u-2|}{u-2}$$



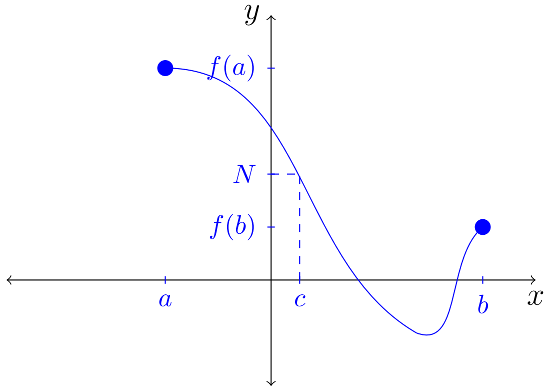
$$\frac{|u-2|}{u-2} = \begin{cases} \frac{u-2}{u-2} = 1 & u > 2 \\ \frac{-(u-2)}{u-2} = -1 & u < 2 \end{cases}$$

$r(u)$ is undefined at $u = 2$ so r is discontinuous there.

$\lim\limits_{u\to 2^+} r(u) = \lim\limits_{u\to 2^+} 1 = 1$
$\lim\limits_{u\to 2^+} r(u) = \lim\limits_{u\to 2^+} -1 = -1$
$\lim\limits_{u\to 2} r(u)$ DNE. But, since the right and left limits exist but differ, $r(u)$ has a jump discontinuity at $u = 2$.

Intermediate Value Theorem (IVT): Suppose f is ____continuous____ on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then:

there exists a number c between a and b such that $f(c) = N$.

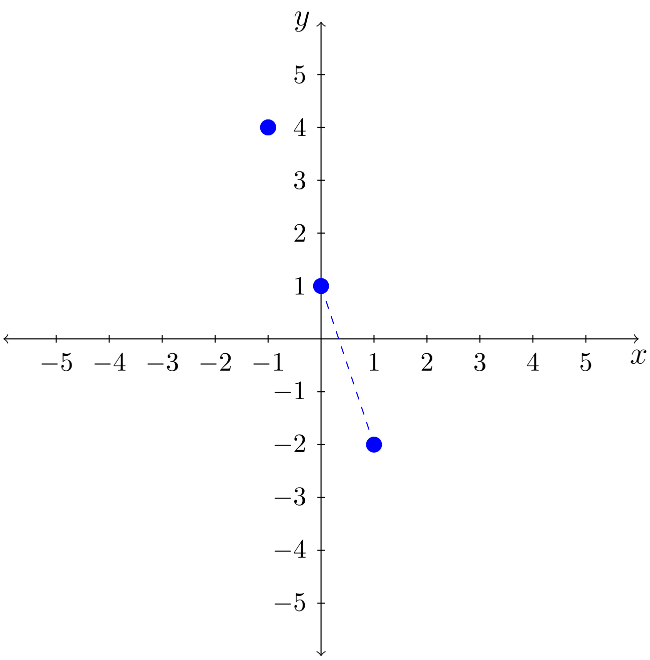


Example: Show that $f(x) = 3x^3 - 6x + 1$ crosses the x-axis somewhere.

Since $f(x)$ is a polynomial, $f(x)$ is continuous. So we can use the Intermediate Value Theorem.

$$f(-1) = 4$$
$$f(0) = 1$$
$$f(1) = -2$$

Since 0 is between 1 and -2, there exists some value c between $x = 0$ and $x = 1$ where $f(c) = 0$.



Note: We could also use the IVT to show (for example):

- $f(c) = \pi$ for some c between $[-1, 0]$.
- $f(x)$ also has roots in $[-2, -1]$ and $[1, 2]$.